A NOTE ON THE DISPERSION RELATION FOR VERY HIGH TEMPERATURE PLASMA

SAROJ K. MAJUMDAR

SAHA INSTITUTE OF NUCLEAR PHYSICS, CALCUTTA

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ABSTRACT. The asymptotic expansion of the function $S_\nu(z)$ which completely describes the plasma dispersion law and introduced by Majumar (1965) has been carried out for very high temperature and low magnetic field. From this expression the well-known results of zero magnetic field case are recovered.

INTRODUCTION

In an earlier work on the dispersion relation for electron plasma in an external magnetic field, (Majumdar, 1965), it was shown that the whole problem of plasma dispersion can be throughly solved in terms of a new function given by

$$S_\nu(Z) = \int_0^\infty e^{i\nu t + Z\cos t}\, dt. \qquad \ldots \quad (1)$$

where

$$\left.\begin{aligned} \nu &= \omega/\omega_c \\ Z &= k_\rho^2 T/m\omega_c^2 \end{aligned}\right\} . \qquad (2)$$

and

Here, $\omega_c = eH_0/mc$ is the cyclotron frequency of the plasma electrons, T is their kinetic temperature, ω and k_ρ are respectively the frequency and wave number (perpendicular to the applied magnetic field $\vec{H}_0$) of the disturbances.

It has been shown (Mazumdar 1965), that the whole problem of dispersion of waves in a magnetic plasma can be solved if one can find the various properties of $S_\nu(Z)$ as a function of ν and Z.

In this report, we wish to study the asymptotic behaviour of the function $S_\nu(Z)$ for very high temperaure and low magnetic field. In that case both ν and Z assume large values, but ν/Z remains small. From the result so obtained, we shall recover the well-known case of zero magnetic field.

ASYMPTOTIC EXPANSION OF $S_\nu(Z)$

It has been discussed (Mazumdar, 1965) that the function $S_\nu(Z)$ can be thought as to belong to the family of Bessel's function. Calling ν as the order and Z as the argument of $S_\nu(Z)$, we need to find the asymptotic expansion for large order and argument. For this purpose, we follow the method of steepest descent. To do this, we first transform the function $S_\nu(Z)$ in a suitable form. Setting $t = iw$ in eqn. (1), we get

$$S_\nu(Z) = i \int_0^{-i\infty} dw e^{Z\cosh w - \nu w}. \qquad .. \quad (3)$$

The contour of the integral (3) runs from 0 to $-\infty$ along the imaginary axis as shown in fig. 1. We divide this contour at all points which correspond to odd

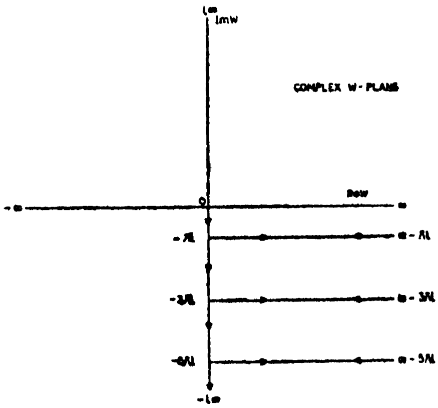

Fig. 1. Contour of integration in eqns, (3) and (4).

multiple of π. At each of these points we add and subtract straight line contours going to $+\infty$ in a direction parallel to the real axis, as shown in fig. 1. The result is that one can write the function $S_\nu(Z)$ in the following form :

$$S_\nu(Z) = i\left[\int_0^{\infty - i\pi} + \int_{\infty - i\pi}^{\infty - 3i\pi} + \int_{\infty - 3i\pi}^{\infty - 5i\pi} + \ldots\right] dw e^{Z\cosh w - \nu w}$$

$$= -i \int_{\infty - i\pi}^{0} e^{Z\cosh w - \nu w}\, dw + i\, \frac{e^{i\pi\nu}}{2\sin\nu\pi} \int_{\infty - i\pi}^{\infty + i\pi} e^{Z\cosh w - \nu w}\, dw. \qquad .. \quad (4)$$

The form of the integrals in (4) is very similar to that of the integral representation of the modified Bessel function of the second kind. Therefore, following Watson (1952), we calculate the asymptotic expansion of $S_\nu(Z)$:

$$S_\nu(Z) = ie^{ZB}\alpha\phi(1/2,\ 3/2;\ \alpha^2 A)$$

$$+ \frac{i}{2} e^{ZB} \sum_{m=0}^{\infty} \frac{a_{2m+1}}{Z^{m+1}} \Gamma(m+1) + \frac{1}{2}(\cot\nu\pi) e^{ZB} \sum_{m=0}^{\infty} \frac{a_{2m}}{Z^{m+\frac{1}{2}}} \Gamma(m+\tfrac{1}{2})$$

$$= S_1 + S_2 + S_3 \quad \text{in the same order.} \qquad (5)$$

In (5), we have used the following :

$$\left.\begin{aligned} \alpha &= \sinh^{-1}\nu/Z \\ A &= (Z/2)\cosh\alpha \\ B &= \cosh\alpha-\alpha\sinh\alpha, \end{aligned}\right\} \qquad (6)$$

the quantities α_m 's are certain coefficients of expansion whose values are tabulated by Watson, (1952). Here we shall use only the a_0 and a_1 terms, which have the following values :

$$\left.\begin{aligned} a_0 &= (-\tfrac{1}{2}\cosh\alpha)^{-\frac{1}{2}} \\ a_1 &= -\frac{2}{3}\frac{\tanh\alpha}{\cosh\alpha} \end{aligned}\right\} \qquad \ldots (7)$$

The function ϕ appearing in eqn. (5) is the confluent hypergeometric function (Erdelyi *et al* 1953.)

Eqn. (5) is the complete expression for the asymptotic expansion of $S_\nu(Z)$. This is a quite complicated expression, but in most cases of interest the summations over m can be replaced by their respective first terms. For example, in the case of high temperature plasma in a low magnetic field, the quantity Z is very large. In that case we are justified in using only the $m = 0$ term ·

LOW MAGNETIC FIELD—HIGH TEMPERATURE PLASMA

In this case, $\nu/Z << 1$. Hence from eqns. (6) and (7) we can write,

$$\alpha_0 \simeq \nu/Z; \quad A \simeq Z/2+\nu^2/4Z; \quad B \simeq 1-\nu^2/2Z$$

$$a_0 \simeq \frac{\sqrt{2}}{i}\left(1-\frac{\nu^2}{4Z^2}\right); \quad a_1 \simeq -\frac{2}{3}\frac{\nu/Z}{1+\nu^2/Z^2}.$$

Using these approximate values, the quantities S_2, S_3 and S_3 whose sum is the asymptotic expression of $S_\nu(Z)$ given in eqn. (5), can be calculated. The result is

$$S_1 \simeq i\frac{\nu}{\eta}\phi(1/2, 3/2; \nu^2/2Z)e^{Z-\nu^2/2Z}$$

$$S_2 \simeq -\frac{i}{3}\frac{\nu}{Z^2}\frac{1}{1+\nu^2/Z^2}e^{Z-\nu^2/2Z}$$

$$S_3 \simeq -\frac{1}{\sqrt{2}}\frac{\Gamma(\frac{1}{2})}{Z^{\frac{1}{2}}}(1-\nu^2/4Z^2)e^{Z-\nu^2/2Z} \qquad (8)$$

and $$S_\nu(Z) = S_1+S_2+S_3.$$

To check the result of the asymptotic expansion, we shall now show that this value of $S_\nu(Z)$ given by eqn. (8) leads to the correct expression of the dispersion relation for zero magnetic field. It has been shown (Mazumdar, 1965) that the dispersion law for plasma wave for zero magnetic field is given by

$$-\omega^2/c^2+I_{xx}=0, \quad .. \quad (9)$$

where

$$I_{xx}=\frac{\omega_p^2}{c^2}\frac{\nu^2}{Z}\,[1+i\nu e^{-z}S_\nu(Z)], \quad .. \quad (10)$$

ω_p is the electron-plasma frequency. Using eqns. (8) and (10) we calculate the limit of I_{xx} for $\omega_c \to 0$, and then use the resulting expression in (9). This gives us the following dispersion relation :

$$-\frac{\omega^2}{c^2}+\frac{\omega_p^2}{c^2}\frac{\nu^2}{Z}\left[-\phi(1,3/2;\,-\nu^2/2Z)+\frac{2}{3}\frac{\nu^2}{Z}\,\phi(2,5/2;\,-\nu^2/2Z)\right]$$

$$+\frac{i}{2^{\frac{1}{2}}}\frac{\omega_p^2}{c^2}\Gamma(1/2)\omega^3\left[\frac{m}{k^2T}\right]^{3/2}e^{-\nu^2/2Z}=0. \quad .. \quad (11)$$

Writing $\omega=\omega_r+i\omega_i$, where ω_r denotes the real part of the frequency and ω_i gives the imaginary part, and using the asymptotic form of the function ϕ, we separate the real and imaginary parts of eqn. (11). The final result is

$$\omega_r^2=\omega_p^2+k^2<V^2>_{av}.$$

and

$$\omega_i=\tfrac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{\omega_p}{(k\lambda_D)^3}\exp-\left(\frac{1}{2k^2\lambda^2_D}\right) \quad .. \quad (12)$$

where $<V^2>^{\frac{1}{2}}_{av}$ is the average electron speed and λ_D is the well-known Debye length. The first equation of (12) is the dispersion relation in the familiar form, whereas the second relation is the well-known expression for Landau damping.

For a non-vanishing magnetic field one can, in a similar manner, calculate the dispersion law for high temperature plasma placed in a low magnetic field. It is also possible to extend the method for other values of the ratio ν/Z and thus may cover a whole range of magnetic field.

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